

A new Contribution to Cosmological Perturbations of some Inflationary Models

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Abstract

We show that there are inflationary models for which perturbations in the energy momentum tensor, which are of second order in the scalar field, cannot be neglected. We first specify the conditions under which the usual first order perturbations are absent. We then analyze classically, the growth and decay of our new type of perturbations for one mode of fluctuations $\delta\phi_k$ in the scalar field. We generalize this analysis, considering the contribution from the whole spectrum of $\delta\phi$ to a given wavelength of geometrical perturbations. Finally, we discuss the evolution of the perturbations during the subsequent radiation dominated era and discuss the resulting spectrum of density fluctuations. In the case of a massless scalar field we find a spectral index $n = 4$. For massive scalar fields we obtain $n = 0$ but the resulting amplitude of fluctuations for inflation around a GUT scale are by far too high. Hence, 'conventional' inflationary models must not be influenced by this new type of perturbations, in order to lead to acceptable perturbations.

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1 Introduction

Inflation was originally proposed [1, 2] to solve some shortcomings of the standard cosmological model. In addition, inflation can provide initial perturbations needed for the formation of the structures observed in our universe. Most inflationary scenarios predict a scale-invariant spectrum of fluctuations [3], known as Harrison–Zel’dovich spectrum [4, 5], which was found to be consistent with observations performed by the COBE satellite [6, 7]. This remarkable agreement motivated many of us in studying the evolution of perturbations in the context of inflation. In particular, since the amplitude of fluctuations poses the stringest constraints on inflationary models, we are interested in analyzing the growth and decay of cosmological perturbations.

To our knowledge, up to now the only perturbations of the energy momentum tensor which have been considered, were first order in $\delta\phi$. (Recently we were told that quantum fluctuations which are second order in $\delta\phi$ have been investigated [8].) On the other hand, since successful inflation requires that $\dot{\phi}$ and $V_{,\phi}$ are small enough, one can envisage situations where $\dot{\phi}$ is of the order of $\delta\dot{\phi}$ and $V_{,\phi} \ll V_{,\phi\phi} \delta\phi$. In this case, all contributions to the perturbation of $T_{\mu\nu}$ are second order in $\delta\phi$. However, it has to be stressed that these perturbations correspond to first order perturbations in both, $T_{\mu\nu}$ and the geometry, and thus they can be analyzed by usual cosmological perturbation theory. This is the issue addressed in this paper.

Our work is organized as follows: In the next section, we analyze the conditions on the potential for which first order perturbations can be neglected. In section 3 we study the growth and decay of geometrical fluctuations induced by a single mode perturbation in $\delta\phi$. We proceed in section 4 by generalizing to a generic spectrum of $\delta\phi$. In this case the whole spectrum contributes to the geometrical perturbations of a given wavelength. In section 5, the evolution of perturbations is followed through the radiation dominated era until second horizon crossing to calculate the resulting power spectrum of density fluctuations. We finally summarize our conclusions.

Since we are only concerned with the very early universe, we neglect spatial curvature. Furthermore, we use the signature $(-, +, +, +)$, such that in conformal time η , the metric is given by $ds^2 = a^2(-d\eta^2 + d\mathbf{x}^2)$. We denote by prime and dot the derivatives with respect to conformal and cosmological time respectively. In what follows, G is the Newtonian gravitational constant, $m_{Pl} = 1/\sqrt{G}$ is the Planck mass ($c = \hbar = 1$) and $H = \dot{a}/a = a'/a^2$ is the Hubble parameter.

2 When are second order perturbations important?

Let us consider a minimally coupled real scalar field ϕ with Lagrangian density

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi). \quad (1)$$

The stress energy tensor of ϕ is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - Lg_{\mu\nu}. \quad (2)$$

During inflation, the energy density of the universe is dominated by the potential term V and thus assuming a homogeneous and isotropic inflaton field ϕ , we have $(1/2)\dot{\phi}^2 \ll V(\phi)$. The equation of motion for $\phi(t)$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi}(\phi) = 0 \quad (3)$$

(where $V_{,\phi}$ denotes the derivative of V with respect to ϕ) and Friedmann's equation, $H^2 = (8\pi G/3)V$, imply $|V_{,\phi}| \ll \sqrt{48\pi}V/m_{Pl}$.

First order perturbations in the energy momentum tensor (2) have the form

$$\dot{\phi}\delta\dot{\phi} \quad \text{or} \quad V_{,\phi}\delta\phi. \quad (4)$$

We now want to study under which conditions the first order perturbations in $\delta\phi$ are negligible, namely, when $\dot{\phi} \sim \delta\dot{\phi}$ and

$$|V_{,\phi}| \ll \max\{|V_{,\phi\phi}\delta\phi|, |\frac{H}{a}\delta\phi'|, \frac{1}{a^2}|\delta\phi''|, \frac{1}{a^2}|\Delta(\delta\phi)|\}, \quad (5)$$

where Δ denotes the usual three dimensional Laplacian. Let us consider the scalar field to be of the form

$$\phi = \phi_0 + \delta\phi, \quad \text{with } \phi_0 = \text{constant and} \quad \delta\phi = \delta\phi_k = mA_k(\eta)e^{ikx}, \quad (6)$$

where we cast the dimension of $\delta\phi$ in the mass parameter m , so that A_k is a dimensionless quantity. In the next section we will appropriately choose the value of m . Provided condition (5) is met, we can then write the equation of motion for $\delta\phi$ as

$$A_k'' + 2HaA_k' + A_k(k^2 + a^2B) = 0, \quad (7)$$

where

$$B \equiv V_{,\phi\phi}(\phi_0).$$

We want to emphasize that the geometrical terms usually present in the above equation (see, *e.g.* [9]) do not contribute in our case, up to first order, since $\dot{\phi}$ and $V_{,\phi}$ are already first order perturbations. The solution of (7) is given by

$$A_k = A_{in}(k)y^{3/2}J_{\pm\nu}(y), \quad \text{with} \quad y \equiv \frac{k}{Ha} \quad \text{and} \quad \nu \equiv \frac{3}{2}\sqrt{1 - \frac{4B}{9H^2}}, \quad (8)$$

where by J_α we denote the Bessel function of order α . As we will show later, for suitable initial conditions $A_{in}(k)$ is actually independent of k , so that it can be absorbed in the definition of m . Thus, without loss of generality we set $A_{in}(k) = 1$. For the space and time derivatives of $\delta\phi$ we then find

$$\delta\dot{\phi} = -\frac{k}{2a}my^{1/2}[3J_{\pm\nu}(y) + yJ_{\pm\nu-1}(y) - yJ_{\pm\nu+1}(y)]e^{ikx}, \quad (9)$$

$$\delta\phi_{,j} = ik_jmy^{3/2}J_{\pm\nu}(y)e^{ikx}. \quad (10)$$

Before proceeding with the discussion of the conditions imposed on the form of the potential V , we will distinguish between super- and sub-horizon perturbations. (Here, as usual, super (sub)- horizon denote larger (smaller) than the Hubble radius respectively.)

For super-horizon perturbations, $\lambda = a/k \gg 1/H$, *i.e.*, $y \ll 1$. For the least decaying mode we then find

$$|\delta\dot{\phi}| = H|(\frac{3}{2} - \nu)\delta\phi|. \quad (11)$$

For $B \geq 0$,

$$|\delta\dot{\phi}|^2 \leq B|\delta\phi|^2 \ll V(\phi_0),$$

where the last inequality sign has to hold in order for the universe to be in an inflationary phase. Therefore, if $|\delta\phi|$ was small initially, it remains small during inflation for arbitrary positive values of B (which is of course also intuitively clear). Furthermore, on super-horizon scales $|\Delta\delta\phi| = k^2|\delta\phi| \ll |\delta\phi''|$ and the last term in (5) can thus be neglected. The limit on $V_{,\phi}$ from (5) then reduces to $|V_{,\phi}(\phi_0)| \ll B|\delta\phi|$. After N e-foldings of inflation $|\delta\phi| \approx m|\exp[N(\nu - 3/2)]|$, so finally the conditions imposed on V are

$$|V_{,\phi}(\phi_0)| \ll Bm \exp(-3N/2), \quad \text{for } B > H^2 \quad (12)$$

$$|V_{,\phi}(\phi_0)| \ll Bm \exp[-N(\frac{3}{2} - \nu)], \quad \text{for } B < H^2 \quad (13)$$

On the other hand, for $B < 0$, A_k grows according to $A_k \propto a^{\nu-3/2}$. For $\delta\dot{\phi}$ not to grow extensively during N e-foldings of inflation, we hence have to require

$$N(\nu - 3/2) \leq 1 \quad \text{or, equivalently,} \quad -H^2/N < B.$$

In that case the conditions on V become

$$|V_{,\phi}(\phi_0)| \ll Bm \quad \text{and} \quad B > \frac{-H^2}{N}. \quad (14)$$

For sub-horizon perturbations, $\lambda = a/k \ll 1/H$, *i.e.*, $y \gg 1$, we have

$$|\delta\dot{\phi}| \approx mHy^2 \approx Hy|\delta\phi| = \frac{k}{a}|\delta\phi| \quad (15)$$

and

$$|\delta\phi_{,j}| \approx k_j|\delta\phi| \approx a|\delta\dot{\phi}|. \quad (16)$$

Therefore, the conditions on the potential V result in

$$|V_{,\phi}(\phi_0)| \ll \max\{B|\delta\phi|, \frac{k^2}{a^2}|\delta\phi|\}. \quad (17)$$

On sub-horizon scales, classical perturbations decay and quantum effects are important, and a quantum analysis has to be performed.

Provided $0 < B \ll H^2$, the $\langle (\delta\phi)^2 \rangle$ can be approximated by

$$\langle (\delta\phi)^2 \rangle \approx \frac{1}{(2\pi)^3} \int_0^\infty \frac{1}{2a^3\sqrt{B + (k/a)^2}} d^3k + \frac{H^2}{4\pi^2} \int_0^{H a} \left(\frac{k}{Ha}\right)^{2B/3H^2} \frac{dk}{k}, \quad (18)$$

which suggests that [10]

$$n_k \equiv \frac{a^3 H^2 \sqrt{B + (k/a)^2}}{2k^3} \left(\frac{k}{Ha}\right)^{2B/3H^2} = \frac{\sqrt{1 + B/(yH)^2}}{2y^2} y^{2B/3H^2}, \quad (19)$$

denotes the number density of ϕ -particles. On sub-horizon scales, $n_k \ll 1$, which means that the quantum analysis is needed. On the other hand, on super-horizon scales, $n_k \gg 1$ and thus our classical analysis is indeed sufficient.

A quantum analysis study in the case of sub-horizon perturbations shows that the main contribution to (18) is given by the second term, which approximately yields

$$\delta\phi(k) \approx \frac{H}{2\pi} \left(\frac{k}{Ha}\right)^{B/3H^2} \approx \frac{H}{2\pi}. \quad (20)$$

A natural choice for m , in the case that $B \ll H^2$, is therefore $m \equiv H/(2\pi)$, leading to

$$\delta\phi(\mathbf{x}) = \int_0^{H^2} \frac{H}{2\pi} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\frac{k}{Ha}\right)^{3/2} J_{\pm\nu}\left(\frac{k}{Ha}\right) d^3k, \quad (21)$$

for super-horizon perturbations.

In the situation where the above imposed conditions on the potential V , namely equations (12), (13), (14) and (17) are satisfied, the perturbations in $T_{\mu\nu}$ are given by

$$\delta T_{\mu\nu} = (\delta\phi)_{,\mu} (\delta\phi)_{,\nu} - \frac{1}{2} g_{\mu\nu} (\delta\phi)_{,\lambda} (\delta\phi)^{,\lambda} + g_{\mu\nu} V_{,\phi\phi} \delta\phi\delta\phi. \quad (22)$$

In the following two sections, we will study the behavior of the geometrical perturbations induced by the above $\delta T_{\mu\nu}$. Before proceeding with this study, we would like to briefly discuss some inflationary models, for which our conditions are met.

- **Old Inflation:** In old inflation [1], the inflaton field ϕ is sitting in a minimum of the potential V , so that $B > 0$ and $V_{,\phi} = 0$, which means that our conditions are satisfied. However, since in this model the phase transition is of first order and, thus, proceeds by bubble nucleation, it has the problem of the so-called graceful exit [11, 12]. This means that the bubble nucleation process is slow with respect to the universal expansion and, therefore, the phase transition can never be completed. On the other hand, if this problem is solved, (*e.g.*, by extended or hyper-extended inflation [13]) then the density fluctuations produced by bubble nucleation will, most probably, be the dominant ones.
- **New Inflation:** In new inflation [14], the inflaton field ϕ is sitting either in a minimum or in a maximum of the potential V , depending on the details of the model. Nevertheless, if $B < 0$, the slow rollover condition requires $|B|$ to be small. Thus, our condition, namely, $B > -H^2/N$, may well be satisfied. In such a case, one might think that the problems usually arising with first order perturbations (*i.e.*, the requirement of extremely flat potential and of a very weakly coupled scalar field) can be avoided. However, a more detailed study has to be performed.
- **Supersymmetric Inflation:** In certain models of supersymmetric inflation [15], both $V_{,\phi}$ and B vanish and our conditions are certainly satisfied. Thus our analysis can indeed be applied.
- **Chaotic Inflation:** In chaotic inflation [16], $V \propto \lambda\phi^n$, so that $V_{,\phi} \sim B\phi_0$ and therefore, $V_{,\phi} \ll B\delta\phi$ requires that $\phi_0 \ll \delta\phi$, which is of course inconsistent with perturbation analysis. As a conclusion, within the context of chaotic inflation our second order perturbations are irrelevant.
- **de Sitter universe:** Regarding an arbitrary scalar field within an inflationary model, our second order perturbation analysis can be applied to study the fluctuations of that field, provided either the potential is very flat, or, the field is sitting in a minimum of a potential, which then can be of an arbitrary form.

3 Perturbation analysis for one mode

In this section we will investigate the evolution of second order perturbations in the scalar field, which are of the specific form $\delta\phi = mA_k(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}$. Since the quantum analysis is only

valid for $B \ll H^2$, we now restrict ourselves to only this case. As we have seen before, the classical analysis applies only for super-horizon perturbations and the value of $\delta\phi$ at horizon crossing is given by $\delta\phi(y=1) = H/(2\pi)$. The solution, Eq. (8), of the classical equation of motion, Eq. (7), for $\delta\phi$ implies that the matching condition at horizon crossing, with $m \equiv H/(2\pi)$, yields

$$A_{in}(k) = 1/J_{\pm\nu}(1) \sim 1 ,$$

which is k -independent. This justifies to drop the prefactor $A_{in}(k)$ in the sequel. We now parametrize the energy momentum tensor Eq. (22) by

$$\delta T_{00} = m^2 k^2 f_\rho, \quad (23)$$

$$\delta T_{0j} = m^2 k f_{v,j}, \quad (24)$$

$$\delta p \equiv \frac{1}{3} \delta T_i^i = \frac{m^2 k^2}{a^2} f_p, \quad (25)$$

$$\delta \tau_{jl} = m^2 k^2 (f_{\pi,jl} - \frac{1}{3} \Delta f_\pi \delta_{jl}), \quad (26)$$

where $\delta \tau_{jl}$ is the traceless part of the perturbed stress tensor. In terms of the function A_k defined above, we obtain

$$f_\rho = \frac{1}{2k^2} [A_k'^2 - A_k^2 (k^2 - a^2 B)] e^{2i\mathbf{k}\cdot\mathbf{x}}, \quad (27)$$

$$f_p = \frac{1}{2k^2} [\frac{1}{3} k^2 A_k^2 + A'^2 - a^2 B A_k^2] e^{2i\mathbf{k}\cdot\mathbf{x}}, \quad (28)$$

$$f_v = \frac{1}{2k} A_k A_k' e^{2i\mathbf{k}\cdot\mathbf{x}}, \quad (29)$$

$$f_\pi = A_k^2 e^{2i\mathbf{k}\cdot\mathbf{x}}. \quad (30)$$

Using the perturbed Einstein equations, we will now relate this energy momentum tensor to the perturbations in the geometry. In the case of scalar perturbations, which are the relevant ones for structure formation, geometrical perturbations can be expressed in terms of the Bardeen potentials Φ , Ψ [17]. In this paper we only analyze scalar perturbations.

Let us now briefly discuss the physical meaning of the Bardeen potentials. They were originally introduced because they are invariant under linearized coordinate transformations, which, in the context of cosmological perturbation theory, are called gauge transformations. As it has been shown [18, 19] (Stewart lemma), gauge invariant quantities are perturbations of tensor fields with either vanishing, or constant, background contribution. Since homogeneous and isotropic universes (Friedmann–Lemaître models) are conformally flat, they have a vanishing Weyl tensor. Thus, according to the Stewart lemma, its perturbations are gauge invariant. The Bardeen potentials are related to the electrical part of the Weyl tensor by

$$E_{ij} \equiv a^{-2} C_{i0j0} = \frac{1}{2} [(\Phi - \Psi)_{|ij} - \frac{1}{3} \Delta(\Phi - \Psi)] ; \quad (31)$$

the magnetic part vanishes for scalar perturbations [20]. The amplitude of perturbations in the geometry is typically given by

$$\mathcal{A} \equiv \max_{\mu\nu\lambda\sigma\alpha\beta} \{ |C_{\mu\nu}^{\lambda\sigma} / R_{\alpha\beta}^{\alpha\beta}| \} \approx \frac{k^2}{a^2 H^2} |\Phi - \Psi| ,$$

where R_{β}^{α} is the Ricci curvature. In the inflationary universe

$$\mathcal{A} \approx y^2(\Phi - \Psi) . \quad (32)$$

Hence, the Bardeen potentials determine the amplitude of perturbations at horizon crossing. On super-horizon scales, where our analysis is relevant, the amplitude of perturbations is constant only if the Bardeen potentials grow as a^2 .

Within the context of this paper, δT_{μ}^{ν} represents the perturbation of the constant tensor field $V(\phi_0)\delta_{\mu}^{\nu}$, thus it is gauge invariant and it can be regarded as a seed perturbation according to [21]. The Bardeen potentials are then given by [21]

$$\Delta\Phi = -\epsilon[f_{\rho}k^2 + 3(a'/a)kf_v] ; \quad (33)$$

$$\Delta(\Phi + \Psi) = -2\epsilon\Delta f_{\pi} , \quad (34)$$

where $\epsilon \equiv 4\pi(m/m_{Pl})^2$ must be much smaller than 1 for linear perturbation analysis to hold. For $B \ll H^2$, ϵ becomes $\epsilon = H^2/(\pi m_{Pl}^2)$. Equations (33) and (34) imply that in our case the Bardeen potentials are

$$\Phi = \frac{\epsilon}{2k^2} \left[A_k'^2 - A_k^2(k^2 - a^2B) + 3\frac{a'}{a}A_kA_k' \right] e^{2i\mathbf{k}\cdot\mathbf{x}} ; \quad (35)$$

$$\Psi = -\frac{\epsilon}{2k^2} \left[A_k'^2 + A_k^2(3k^2 + a^2B) + 3\frac{a'}{a}A_kA_k' \right] e^{2i\mathbf{k}\cdot\mathbf{x}} . \quad (36)$$

On super-horizon scales ($y \ll 1$), A_k given by Eq. (8) reduces to

$$A_k = \frac{2^{\nu}}{\Gamma(1-\nu)} y^{3/2-\nu} . \quad (37)$$

Inserting this result in equations (35) and (36), we find

$$\Phi = \frac{-2^{2\nu-1}\epsilon}{[\Gamma(1-\nu)]^2} y^{3-2\nu} e^{2i\mathbf{k}\cdot\mathbf{x}} ; \quad (38)$$

$$\Psi = \frac{-3 \cdot 2^{2\nu-1}\epsilon}{[\Gamma(1-\nu)]^2} y^{3-2\nu} e^{2i\mathbf{k}\cdot\mathbf{x}} . \quad (39)$$

For the relevant regime, $B \ll H^2$ and the above expressions lead to

$$3\Phi = \Psi = \frac{-3}{\pi} \epsilon e^{2i\mathbf{k}\cdot\mathbf{x}} . \quad (40)$$

The Bardeen potentials are constant in time meaning that the amplitude of curvature perturbations decays exponentially.

We shall see in Section 5, how the amplitude ϵ of Ψ is connected with observed fluctuations in the cosmic microwave background and what its value has to be due to a normalization, e.g. according to the COBE quadrupole.

4 Perturbation analysis for the whole spectrum

So far we have only considered the contribution of one wave vector \mathbf{k} . However, since the Bardeen potentials are quadratic in $\delta\phi$, to a given wave vector \mathbf{k}_1 of the (Φ, Ψ) spectrum, there is a contribution from all wave vectors \mathbf{k} of $\delta\phi$. We therefore address in this section, the contribution of the whole spectrum of \mathbf{k} in the evolution of the Bardeen potentials.

We use the general ansatz

$$\delta\phi = mA(\mathbf{x}, \eta).$$

The f_ρ, f_v, f_π functions, which enter in the Bardeen potentials, become

$$f_\pi(\mathbf{x}) = A^2 ; \quad (41)$$

$$\nabla_j f_v(\mathbf{x}) = lA'\nabla_j A ; \quad (42)$$

$$f_\rho(\mathbf{x}) = l^2[A'^2 + (\nabla A)^2 + a^2BA^2]/2 , \quad (43)$$

where l denotes an arbitrary length scale, for example $l = 1/k_1$. To find the spectrum of the above functions we now perform a Fourier transform. To obtain a finite result, we have to introduce a cutoff. Taking into account that the fluctuations $A(k)$ are distributed with uncorrelated phases, it makes sense to neglect contributions from wavelengths $\lambda < \lambda_1$, i.e., to choose the cutoff k_1 and correspondingly the normalization volume k_1^3 . Using the convolution theorem, we then obtain

$$f_\pi(\mathbf{k}_1) \approx \frac{1}{(2\pi)^{3/2}k_1^3} \int_0^{k_1} d^3k A(|\mathbf{k}|)A(|\mathbf{k}_1 - \mathbf{k}|) ; \quad (44)$$

$$f_v(\mathbf{k}_1) \approx \frac{1}{(2\pi)^{3/2}k_1^4} \int_0^{k_1} d^3k A'(|\mathbf{k}|)A(|\mathbf{k}_1 - \mathbf{k}|)\mathbf{k}_1 \cdot (\mathbf{k}_1 - \mathbf{k}) ; \quad (45)$$

$$f_\rho(\mathbf{k}_1) \approx \frac{1}{2(2\pi)^{3/2}k_1^5} \int_0^{k_1} d^3k [A'(|\mathbf{k}|)A'(|\mathbf{k}_1 - \mathbf{k}|) + (\mathbf{k} \cdot (\mathbf{k}_1 - \mathbf{k}) + a^2B)A(|\mathbf{k}|)A(|\mathbf{k}_1 - \mathbf{k}|)] . \quad (46)$$

With

$$A(|\mathbf{k}|) = \frac{2^\nu}{\Gamma(1-\nu)} y^{3/2-\nu},$$

the above integrals finally yield

$$f_\pi(\mathbf{k}_1) \approx \sqrt{\frac{8}{9\pi^3}} ; \quad (47)$$

$$f_v(\mathbf{k}_1) \approx -\frac{B}{3H^2} \approx 0 ; \quad (48)$$

$$f_\rho(\mathbf{k}_1) \approx f_\pi \left[\frac{B^2}{9H^4} + \frac{3}{5} + \frac{B}{H^2} \right] \approx \frac{3f_\pi}{5} , \quad (49)$$

where $y_1 = k_1/(Ha)$. Inserting these results in equations (33 and 34), we find for the Bardeen potentials

$$\Phi \approx -\Psi \approx \epsilon \frac{\sqrt{2\pi}}{5} . \quad (50)$$

This result shows that the Bardeen potentials remain constant during the inflationary period and the geometrical perturbation amplitude \mathcal{A} , introduced in the last section, thus decays like $1/a^2$. However, the relevant physical quantity for structure formation, is the change of $\delta\rho/\rho$ between the first and second horizon crossing. At horizon crossing, $\delta\rho/\rho$ is well approximated by the Bardeen potentials [9]. Hence, we will now study the evolution of the Bardeen potentials during the radiation dominated era, until the time of second horizon crossing.

5 The radiation dominated era and second horizon crossing

In this section H without argument denotes the value of the Hubble parameter during inflation. The value of a quantity at the end of inflation is indicated by the subscript $_{end}$ and at present time by a subscript $_0$.

5.1 The Bardeen potentials at second horizon crossing

According to the general equation of motion for $\delta\phi$, (7), the evolution of the perturbation amplitude A_k in the radiation dominated regime is given by

$$\frac{d^2 A_k}{dy^2} + \frac{2}{y} \frac{dA_k}{dy} + (1 + \alpha y^2) A_k = 0, \quad (51)$$

where $y = k/(aH(y)) = k\eta$ in the radiation dominated epoch, and $\alpha \equiv a^2 B/(k^2 \eta)^2 = \text{const.}$ Since at the end of inflation $\eta = 1/(aH)$ to a very good accuracy, we find

$$\alpha = \frac{B}{H^2 y_{end}^4}.$$

Here y_{end} is the value of y at the end of inflation and H denotes the Hubble parameter during the inflationary period. For $B = 0$, (51) is exactly solved in terms of Bessel functions and the 'growing mode' is given by

$$A_k(y) = \frac{1}{\sqrt{y}} J_{1/2}(y).$$

This solution is still approximately valid as long as $\alpha y^2 < 1$. To require $\alpha y_{end}^2 < 1$ for all physically interesting scales, $H a_0 \leq k \leq H_0 a_0$ yields in terms of the mass parameter B

$$B \ll z_{end}^2 H_0^2 \sim (10^{-6} \text{eV})^2 \cdot (T_{end}/10^{15} \text{GeV}). \quad (52)$$

Obsevationally, such a small mass can not be distinguished from $B = 0$. The general solution to (51) at $y = 1$ can be found by exponentiating the matrix

$$\begin{aligned} M(y, y_{end}) &= \int_{y_{end}}^y \begin{pmatrix} 0 & 1 \\ -(1 + \alpha y^2) & -2/y \end{pmatrix} dy \\ &= \begin{pmatrix} 0 & y - y_{end} \\ -(y - y_{end} + \alpha/3(y^3 - y_{end}^3)) & 2 \ln(y_{end}/y) \end{pmatrix}, \end{aligned}$$

which determines the solution according to

$$\begin{pmatrix} A(y) \\ A'(y) \end{pmatrix} = \exp(M(y, y_{end})) \begin{pmatrix} A(y_{end}) \\ A'(y_{end}) \end{pmatrix}.$$

Unfortunately M cannot be diagonalized; but explicit exponentiation (by expansion in power series) yields a simple, summable result if $|M_{21}| \gg |M_{12}|, |M_{22}|$, which is valid for $\alpha \gg 1$ and $y \gg y_{end}$. In the limit $\alpha \gg 1$ we find

$$\exp(M(1, y_{end})) \approx (1 + \ln(y_{end})) \begin{pmatrix} \cos \sqrt{\alpha/3} & \sqrt{3/\alpha} \sin \sqrt{\alpha/3} \\ -\sqrt{\alpha/3} \sin \sqrt{\alpha/3} & \cos \sqrt{\alpha/3} \end{pmatrix}.$$

For each element in $\exp(M)$ we have kept only the highest power in α . Using the exact solution during the inflationary epoch, eqn. (37), which yields

$$A'(y_{end}) = \frac{\alpha}{3} y_{end}^3 A(y_{end}) , \quad (53)$$

we obtain for A_k, A'_k at $y = 1$

$$\begin{aligned} A_k(1) &= (1 + \ln(y_{end})) [\cos \sqrt{\alpha/3} + y_{end}^3 \sqrt{\alpha/3} \sin \sqrt{\alpha/3}] A_k(y_{end}) \\ A'_k(1) &= (1 + \ln(y_{end})) [(\alpha/3) y_{end}^3 \cos \sqrt{\alpha/3} - \sqrt{\alpha/3} \sin \sqrt{\alpha/3}] A_k(y_{end}) . \end{aligned}$$

In the radiation dominated era eqn. (36) for the Bardeen potential Ψ yields

$$\Psi_{rad}(y) = -(\epsilon/2) \left[\left(\frac{dA}{dy} \right)^2 + A^2(3 + \alpha y^2) + (3/y) A \frac{dA}{dy} \right] . \quad (54)$$

For $\alpha \gg 1$ (which implies $y_{end} \ll 1$) this yields

$$\Psi_{rad}(y = 1) \approx -\frac{\epsilon\alpha}{3} A^2(y_{end}) . \quad (55)$$

Equation (54) at $y = y_{end}$ with the initial condition (53) leads for $\alpha y_{end}^2 \gg 1$ to

$$\Psi_{rad}(y_{end}) \approx -\epsilon \alpha y_{end}^2 A^2(y_{end}) \propto y_{end}^{-2} .$$

On the other hand, from the inflationary universe calculation we have $\Psi_{inf}(y_{end}) = \epsilon = \text{const.}$ Assuming now, that the Bardeen potentials do not jump when the universe enters the radiation dominated phase, Ψ has changed since the end of inflation until second horizon crossing by a factor

$$g(k) = \Psi_{rad}(1)/\Psi_{rad}(y_{end}) \approx y_{end}^{-2}/3 . \quad (56)$$

This yields for the final Bardeen potential at second horizon crossing

$$\Psi(\eta_k) \approx -\frac{\epsilon}{3y_{end}^2} = -\frac{\epsilon(Ha_{end})^2}{3k^2} \quad \text{for} \quad \alpha = B/(H^2 y_{end}^4) = BH^2 a_{end}^4/k^4 \gg 1 , \quad (57)$$

where η_k denotes the time of second horizon crossing of wave number k , *i.e.* $\eta_k = 1/k$ and $\epsilon = H^2/(\pi m_{Pl}^2)$. In the case $B = 0$ we find $\Psi_{rad} \approx \text{const.}$, independent of y_{end} . This leads to a growth factor of $g(k) \approx 1$, and thus

$$\Psi(\eta_k) \approx -\epsilon \quad \text{for} \quad B = 0 . \quad (58)$$

In the intermediate regime, $\mathcal{O}(\alpha) \sim 1$, we have solved (51) numerically. The solutions A_k for some values of α are represented in Fig. 1.

In Fig. 2 we plot the value of $\Psi(\eta_k)$ in units of ϵ as a function of physical wave number k/a in units of H (In other words, we plot Ψ as a function of y_{end}). We chose the parameter value $B/H^2 = 0.01$. For small wave numbers, $\alpha \gg 1$, the amplitude of Ψ decays like k^{-2} until approximately $\alpha y_{end}^2 = BH^2/(k/a_{end})^2 \sim 1$. At $(k/a_{end})^4 \sim BH^2$, *i.e.* $\alpha \sim 1$ the potential bends over to a constant.

This results remain qualitatively valid also in the matter dominated era which we thus shall not treat separately.

5.2 The power spectrum

Let us now briefly clarify what power spectrum results from the findings of the last subsection. Since the explanations of spectral index and scale invariance found in the literature are often somewhat confusing, we shall be rather explicit in this paragraph.

The power spectrum is defined by

$$P(k, \eta) = |\delta_k(\eta)|^2 ,$$

where $\delta_k(\eta)$ is the Fourier transform of the density fluctuation $(\delta\rho/\rho)(x, \eta)$. Clearly for perturbations which are larger than the size of the horizon, $k < 1/\eta$, $P(k, \eta)$ is gauge dependent. But we are mainly interested in $P(k, \eta_0)$ on scales $k > 1/\eta_0$ and $a_0/k \approx \lambda_0 \geq 10\text{Mpc}$, *i.e.*, scales which are small enough to be observable today and large enough, so that they are probably not severely affected by nonlinear clustering. The spectrum $P(k, \eta_0)$ is called scale invariant, or a Harrison–Zel’dovich spectrum, if the variance of the mass perturbation on horizon scale $l_H = a\eta$ is constant, time independent:

$$\langle (\frac{\delta M}{M})_{l_H}^2 \rangle = \text{const.}$$

Here $\langle \rangle$ denotes statistical average over many realizations of perturbed universes with identical statistical properties, and we assume that it can be replaced by a spatial average.

$$(\delta M)_{l_H} = \int_{V_H} \delta\rho d^3x = \rho \int_{V_H} (\delta\rho/\rho) d^3x \approx \rho V_H \int_0^{k_H} \delta_k e^{ikx} d^3k .$$

Here we have assumed that perturbations on scales smaller than l_H ($k > k_H = 1/\eta$) average to zero due to the integration over V_H , and that perturbations on scales larger than l_H ($k < k_H$) are approximately constant in V_H , such that integration over V_H just gives rise to a factor V_H . Using $M = \rho V_H$, we obtain

$$(\delta M/M)_{l_H} \approx \int_0^{k_H} \delta_k e^{ikx} d^3k , \tag{59}$$

such that

$$\begin{aligned} \langle (\delta M/M)_{l_H}^2 \rangle &\approx \int d^3x \int_0^{k_H} d^3k \int_0^{k_H} d^3k' \delta_k \delta_{k'} e^{i(k+k')x} \\ &= \int_0^{k_H} |\delta_k|^2 d^3k = \int_0^{k_H} P(k, \eta) d^3k , \end{aligned}$$

For the first equal sign we have evaluated the spatial integration leading to a δ -distribution and the k' integration using $\delta_{-k} = \delta_k^*$. If we now assume $P(k, \eta)$ behaves like a power law in k , $P(k, \eta) \propto k^m$, scale invariance requires $m = -3$, such that $\delta_k(\eta_k) \propto k^{-3/2}$, where as before η_k denotes the conformal time at which the scale k^{-1} enters the horizon, *i.e.* $\eta_k = k^{-1}$. (We should not be concerned about the logarithmic singularity for $k \rightarrow 0$. We anyway have to introduce a low end cutoff to (59) at about k_{H_0} , since perturbations on scales larger than the present Hubble radius cannot be distinguished from background contributions.)

For a scale invariant spectrum we thus have to require

$$\delta_k(\eta) \propto k^{-3/2} \quad \text{on super horizon scales.}$$

Since we always can choose a gauge such that $\delta_k(\eta) \approx (\eta k)^2 \Psi_k(\eta)$, this yields

$$\Psi_k(\eta_k) \approx \delta_k(\eta_k) \propto k^{-3/2}.$$

As is well known from linear perturbation theory (see e.g. [24]), matter perturbations on scales which enter the horizon during the matter dominated era subsequently start growing according to $\delta_k(\eta) \propto a(\eta) \propto \eta^2$. Perturbations on smaller scales which enter the horizon already in the radiation dominated era, remain approximately constant until the time of equal matter and radiation, η_{eq} , and then also grow according to $\delta \propto a$. We therefore end up with the following behavior of density fluctuations at late times $\eta > \eta_{eq}$:

$$\delta_k(\eta) \approx \begin{cases} Dk^{1/2}\eta^2 & \text{for } k^{-1} > \eta_{eq} \\ Dk^{-3/2}(\eta/\eta_{eq})^2 & \text{for } k^{-1} < \eta_{eq}, \end{cases}$$

or

$$P(\eta, k) \approx \frac{D^2 k \eta^4}{(1 + (k/k_{eq})^2)^2}, \quad \text{where } k_{eq}^{-1} = \eta_{eq}.$$

D is a constant of proportionality which is determined by the value of δ_k at horizon crossing, $\delta_k(\eta_k) = Dk^{-3/2}$. On large scale we then find

$$P(\eta_0, k) \propto k^n, \quad \text{with } n = 1.$$

The power n in this law is the so called spectral index.

We have thus seen, a power spectrum $P(\eta_0, k) \propto k^n$ on large scales leads to the behavior

$$\Psi(\eta_k, k) \propto k^{n/2-2}. \quad (60)$$

Therefore, the behavior of $\Psi(\eta_k, k)$ in our model yields spectral indices

$$n = \begin{cases} 4 & \text{for } B = 0 \\ 0 & \text{for } B \gg (10 Mpc)^{-2} \approx (10^{-30} eV)^2, \end{cases} \quad (61)$$

As is discussed in detail in [22], the COBE experiment is still compatible with spectral indices within the bounds $-2 \leq n \leq 2$, with a peak in the likelihood around $n \approx 1.3$. The case $B = 0$ is therefore excluded by the COBE DMR limits on the spectral index. The possibility $B \neq 0$ cannot be ruled out from the COBE data alone. The relatively strong clustering of galaxies on large scales (see e.g. [23]) even strongly disfavors a large spectral index, but this always depends on the matter model assumed (CDM is only consistent with $n \leq 1$, whereas HDM prefers larger spectral indices).

Let us now investigate the amplitude of the induced perturbations. The value of $\epsilon = H^2/(\pi m_{Pl}^2)$ can be related to the amplitude of the quadrupole of microwave background anisotropies. C_2 can quite generally be determined by the density perturbations today according to (see [25])

$$C_2 \approx \int_0^\infty |\delta_k(\eta_0)|^2 (j_2(y)/y)^2 dy, \quad (62)$$

with $y = k\eta_0$. Using $\delta_k(\eta_0) = \epsilon y_{end}^{-2} (\eta_0 k)^2 / 3 = (\epsilon/3)(\eta_0/\eta_{end})^2$ for large enough scales which are relevant for the quadrupole anisotropy, we obtain in the case $B \neq 0$

$$C_2 \approx \epsilon^2 \frac{z_{end}^4}{3000 z_{eq}^2}. \quad (63)$$

With the COBE quadrupole of $C_2 \approx 3 \cdot 10^{-11}$ this yields $H \approx 0.02 GeV$. A scalar field mass of $m = \sqrt{B} \ll H$, certainly incompatible with the standard model of particle physics which is well tested at these low energies.

For $B = 0$ one finds $C_2 \approx \epsilon^2 = (H/m_{pl})^2$, leading to $H \approx 10^{16} GeV$, a typical GUT scale, but this possibility is already ruled out due to its large spectral index. Certainly, as soon as reliable intermediate measurements of the cosmic microwave background anisotropies are available, the spectral index n will become even more constrained.

6 Conclusions

In this work we have investigated inflationary models in which the perturbations of second order in $\delta\phi$ are the dominant ones. We have studied the evolution of these perturbations. Depending on the mass of the scalar field, we found that the resulting density fluctuations have spectral index $n = 4$ for massless fields and $n = 0$ for massive scalar fields, the second of which is compatible with the COBE DMR results. However, the quadrupole amplitudes for the two cases are completely different:

$$C_2 \approx \begin{cases} (H/1GeV)^6 & \text{for } B \neq 0 \\ (H/m_{pl})^2 & \text{for } B = 0 \end{cases} \quad (64)$$

In the case $B \neq 0$, for an inflationary model to yield small enough perturbations, one therefore must make sure that the perturbations of second order in the scalar field discussed in this work are substantially suppressed. Otherwise, due to the enormous increase $\propto y_{end}^{-2}$ of the Bardeen potentials during the radiation era, they will soon dominate and spoil the perturbation theory.

For 'intermediate' masses, $m = \sqrt{B} \sim 10^{-4} eV$, the spectral index can turn out to be of order $n \sim 0 - 2$ and the amplitude $C_2 \sim \epsilon^2$. (When the relevant physical scales, $100 Mpc \leq k^{-1} \leq 1000 Mpc$ happen to lay at the bending point of Ψ shown in Fig. 2.) But this requires severe fine tuning of the scalar field mass and is thus not investigated any further.

We have not followed through the reheating period, the transition from the inflationary to the radiation dominated phase. We just assumed that perturbations induced during this short time can be neglected on super horizon scales, such that the Bardeen potentials pass continuously over the transition. If our scalar field represents the inflaton field, this assumption has to be justified by further work. But if we regard it as a scalar field in an external de Sitter universe which later becomes radiation dominated, our analysis is certainly valid.

In the radiation era we have only analyzed one mode of perturbations of the scalar field. In principle we would have to perform the convolution presented in Section 4 to obtain the entire contribution to a given mode k_1 of the Bardeen potentials which are quadratic in the scalar field. Repeating the calculation outlined in Section 4, one immediately sees if $A_k \sim const.$ (which is the case for $B = 0$) also the Bardeen potentials turn out to be constant and of the same order of magnitude as for one mode. On the other hand, for $B \neq 0$ A_k is a rapidly oscillating function of $\sqrt{\alpha} \propto 1/k^2$ and thus the main contribution comes from the wave number pair $k = k_1/2$ and we do not expect the behavior of the spectrum to differ substantially from the one mode result. Therefore, we expect the results (61) and (63) obtained in a one mode analysis to remain valid for the full spectrum of fluctuations.

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Figure Captions

Fig. 1 The amplitude of the scalar field, A_k is shown as a function of y for 3 characteristic values of α .

Solid line: $\alpha = 10^5$, with $B/H^2 = 10^{-4}$, such that $y_{end} = 0.056$.

Dotted line: $\alpha = 120$, with $B/H^2 = 10^{-4}$, such that $y_{end} = 0.03$.

Dashed line: $\alpha = 1$, with $B/H^2 = 10^{-4}$, such that $y_{end} = 0.1$.

Fig. 2 The amplitude of the Bardeen potential Ψ at second horizon crossing, $k = \eta^{-1}$ is shown in arbitrary units as function of $y_{end} = k/(a_{end}H)$. The dashed line shows the k^{-2} behavior. In this example $B/H^2 = 10^{-4}$, so that $\alpha y_{end}^2 = 1$ for $y_{end} = 0.01$. The calculated curve even becomes steeper at somewhat larger values of αy^2 and then, around $\alpha = 1$ (corresponding to $y_{end} = 0.1$), bends over to a constant.

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